Computer Science Department

TECHNICAL REPORT

"O (log$n$) Parallel Time Intersection and Union Algorithms for a set of Planar Discs"

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Technical Report #288
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Abstract

We present here O(logn) time parallel algorithms, on the CREW PRAM model of parallel computation, for determining all the arcs and their sequence in the boundary of the intersection or the union of n circular discs in the plane. Here, the intersection of n circular discs is the part of the plane which belongs to all of them. Our techniques use $O(n^3)$ processors. Our algorithms take O(loglog n) time in the CRCW PRAM model of parallel computation.

We also present matching lower bounds for the parallel time of the intersection and union problems. Our algorithms are based on an interesting characterization of the boundary of the intersection and the union. We also show that the boundary of the intersection of n circular discs in the plane, consists of at most 2n-2 arcs, for any $n \geq 2$.

Introduction

The problem of computing the intersection or union of a set of planar objects has been given some attention in the past (see e.g. [Bentley, Ottman, 79], [Sharir, 83]). When the objects are circular discs or boxes then the problem relates to that of quick computation of intersection or union of relations in database applications.

Related problems, such as estimating the area of the union of many discs in the plane, were first posed by [Shamos, 78]. Monte-Carlo techniques for computing the volume of the union of n spheres in k dimensions were given in [Spirakis, 85]. They run in time O(nk) and are based on a method developed by [Karp, Luby, 83] for estimation of the failure probability of an n-component system. Also, [Sharir, 83] shows how to construct the Voronoi diagram of a set of n circular bodies, in time $O(n\log^2 n)$ and this can be used for detecting intersections between the circular bodies. [Brown, 79], shows O(nlogn) <u>sequential</u> algorithms, based on inversion transforms, which compute the (boundam of) the intersection and union of n circles in the plane (see also[Mehlhorn, 1984]).

We consider here the problem of computing all the arcs and their sequence in the boundary of the intersection (or the union) of n circular discs in the plane. We first provide a nice characterization of those boundaries. This leads to an O(logn) parallel time algorithm, running on the CREW PRAM model of parallel computation. Our algorithm takes O(loglogn) time on a CRCW PRAM. It needs n^3 processors. Since O(nlogn) sequential algorithms for computing the intersection or union are known, our bound on the number of processors is by no means the best possible, (see [Brown, 79]). We also show that the boundary of the intersection of n circular discs in the plane, consists of at most 2n-2 arcs, for any $n \geq 2$. To the best of our knowledge, this is the first set of results concerning the <u>parallel</u> computation of the boundaries of union and intersection of many discs.

3. Notation and definitions

A circular disk $\overline{K}$, of radious r and center (x,y) is the set $\{(x',y'):(x-x')^2+(y-y')^2\leq r^2\}$. Its interior, K, is the set $\{(x',y'):(x-x')^2+(y-y')^2< r^2\}$ and its circumference, C, is the set $\{(x',y'):(x-x')^2+(y-y')^2=r^2\}$. It follows that $\overline{K}=K\cup C$ and that $K\cap C=\emptyset$.

All arcs in C are directed counterclockwise (in this paper). A <u>closed</u> arc (which includes the end-points a,b) is denoted [a,b]. Parentheses are used for open and semi-open arcs. So, the complement of [a,b], is $C-[a,b]=(b,a)=[a,b]^c$. An one point arc [a,a]= a has (a,a) as its complement, hence $C=\{a\}\cup(a,a)=[a,a)=(a,a]$.

The ordering of points on the circumference C of K, in the counterclockwise direction, depends on the point of C used as an origin. In Figure 1, q preceeds r if p is the origin while q follows r if p' is the origin.

<u>Definition</u> We say that point x in C preceeds point y in C with respect to p in C iff $x\in[p,y]$. We denote this by $p:\langle x,y\rangle$

More generally, let L be a finite list of points in C which may include p and may also include multiple points.

<u>Definition</u> We say that L is ordered with respect to $p\in C$, and we denote this by $L=p\langle x_1,\ldots,x_k\rangle$ iff $p:\langle x_i,x_j\rangle$ for $i\leq j$ and $i,j\leq k$.

<u>Definition</u> Let L be a finite list of points in C. Consider L as a set i.e. drop repeated points. We denote by Right (x;L) the point r of C with the property

$$[x,r]\cap L=\{r\}$$

Also, we denote by Left (x;L) the point h of C such that

$$[h,x]\cap L=\{h\}$$

<u>Note</u> If $x\in L$ then Left (x;L)=Right (x;L)=x. If $L=x\langle x_1,x_2,\ldots,x_k\rangle$ and all x_i's are distinct, then Right(x;L)=x_1 and Left(x;L)=x_k

Now, we are given n discs, $\overline{K}_i$, $i=1,\ldots,n$ (with circumferences C_i, $i=1,\ldots,n$). Without loss of generality assume $\overline{K}_i \neq \overline{K}_j$ for $i \neq j$.

Definition

The intersection of $\overline{K}_1,\ldots,\overline{K}_n$ is the set $I=\bigcap_{i=1}^{n} \overline{K}_i$

The union of $\overline{K}_1,\ldots,\overline{K}_n$ is the set $\Sigma=\bigcup_{i=1}^{n} \overline{K}_i$.

Definition

Let us fix i. If $\overline{K}_i \cap \overline{K}_j \neq \emptyset$ for some j, then consider the unique arc $C_i \cap \overline{K}_j \neq \emptyset$, which is the part of C_i that lies in $\overline{K}_j$. Let b_{ij} be the beginning and e_{ij} the end of this arc (ordered counterclockwise on C_i).

Thus $[b_{ij},e_{ij}]=C_i \cap \overline{K}_j \subseteq C_i$

Clearly (Fig. 2) $b_{ji}=e_{ij}$. The first index is always meant with respect to the circumference which the arc belongs to.

Thus, $[b_{ji},e_{ji}]$ is an arc in C_j, while $[e_{ij},b_{ij}]$ is an arc in C_i, both having the same endpoints.

The intersection

$$\Lambda_i = \bigcap_{j\neq i} (C_i \cap \overline{K}_j) = \bigcap_{j\neq i} [b_{ij},e_{ij}] \subseteq C_i \qquad (*)$$

is the part of C_i which lies within all $\overline{K}_j$, $j\neq i$. Λ_i could be empty or consist of one or more arcs.

Similarly, the union

$$V_i = \bigcup_{j\neq i} (C_i \cap \overline{K}_j) \qquad \text{is such that}$$

its complement, with respect to C_i, i.e.

$$V_i^c = \bigcap_{j\neq i} (C_i \cap \overline{K}_j)^c = \bigcap_{j\neq i} (e_{ij},b_{ij}) \qquad (**)$$

is the part of C_i that lies outside all $\overline{K}_j$, $j\neq i$.

4. Characterization of the boundaries

Intuition says that Λ_i (respectively V_i^c) must be the contribution of C_i to the boundary of $\bigcap_{i=1}^{n} \bar{K}_i$ (respectively of $\bigcup_{i=1}^{n} \bar{K}_i$). The following theorem proves this:

<u>Theorem 4.1</u>

(a) The boundary of the intersection I is $\bigcup_{i=1}^{n} \Lambda_i$.

(b) The boundary of the union Σ is $\bigcup_{i=1}^{n} V_i^c$.

<u>Proof</u>

We show only part (a) here. Part (b) is similar.

It is enough to show that $I = (\bigcap_{i=1}^{n} K_i) \cup (\bigcup_{i=1}^{n} \Lambda_i)$

(i) Let I' be the set $(\bigcap_{i=1}^{n} K_i) \cup (\bigcup_{i=1}^{n} \Lambda_i)$.

Let $x \in I$. This means $x \in (K_i \cup C_i)\ \forall i=1,\dots,n$.

If $x \in K_i$ for all $i=1,\dots,n$ then clearly $x \in I'$

Now consider $x \in C_m$ for some m.

Since also $x \in I$, we get that

$$x \in (I \cap C_m) \quad \text{i.e.}$$

$$x \in (C_m \cap (\bigcap_{i=1}^{n} \bar{K}_i)) \Rightarrow x \in \Lambda_m \Rightarrow x \in I'$$

(ii) Conversely, assume $x \in I'$.

I.e. $x \in (\bigcap_{i=1}^{n} K_i) \cup (\bigcup_{i=1}^{n} \Lambda_i)$

So, either $x \in (\bigcap_{i=1}^{n} K_i)$ or $x \in \Lambda_m$ for some m

If $x \in (\bigcap_{i=1}^{n} K_i)$ Then $x \in I$, since $\bigcap_{i=1}^{n} K_i \leq \bigcap_{i=1}^{n} \bar{K}_i$

If $x \in \Lambda_m$ for some m then $x \in \bigcap_{j \neq m} (C_m \cap \bar{K}_j)$

But $C_m \cap \overline{K}_j \subseteq \overline{K}_j$ and $C_m \subseteq \overline{K}_m$

So $x \in \bigcap_{j=1}^{n} \overline{K}_j = I$

QED

Theorem 4.1 is useful to a characterization of the sets Λ_i, V_i^c.

<u>Definition</u>

We define the lists B_i, E_i, L_i of points of C_i to be

$B_i := (b_{ij},\ j \neq i,\ j=1,\ldots,n)$

$E_i := (e_{ij},\ j \neq i,\ j=1,\ldots,n)$

$L_i := (b_{ij},\ e_{ij},\ j \neq i,\ j=1,\ldots,n)$

All lists may contain multiple points.

<u>Note</u> If $C_i \cap \overline{K}_j = \emptyset$ for all $j \neq i$ then all lists are empty.

If $C_i \cap \overline{K}_m = \emptyset$ for some m, then $\Lambda_i = \emptyset$, $V_i^c = C_i$.

<u>Definition</u>

$\beta(i)$ is defined to be the set of indices j satisfying

$b_{ij} : \langle e_{ik}, b_{ik} \rangle$ for all $k \neq i, j$ (with $b_{ik} \neq b_{ij}$)

Similarly,

$\varepsilon(i)$ is defined to be the set of indices j satisfying

$e_{ij} : \langle b_{ik}, e_{ik} \rangle$ for all $k \neq i, j$ (with $e_{ik} \neq e_{ij}$)

The following two theorems compose the main result of our paper and lead to a fast parallel algorithm for computing the boundaries of I and Σ:

<u>Theorem 4.2</u>

(a) $\Lambda_i = \bigcup_{j \in \beta(i)} [b_{ij}, \text{Right } (b_{ij}; E_i)]$

(b) $V_i^c = \bigcup_{j \in \varepsilon(i)} [e_{ij}, \text{Right } (e_{ij}; B_i)]$

<u>Proof</u>

<u>Part a</u>

<u>1. $\Lambda_i \subseteq \bigcup_{j \in \beta(i)} [b_{ij}, \text{Right } (b_{ij}, E_i)]$</u>

Assume $x \in \Lambda_i$. Then there exists a j such that b_{ij}=Left$(x;B_i)$
Let e_{ir} be the Right $(b_{ij};E_i)$.
We will show that $x \in [b_{ij},e_{ir}]$
We consider two cases

<u>Case 1</u> If $b_{ij}=x$ then clearly $x \in [b_{ij},e_{ir}]$

<u>Case 2</u> $b_{ij} \neq x$. Then $x \neq b_{ik}$ for all k, by the definition of Left.
Also, (for same reason) $b_{ir} \in (x,b_{ij}]$.
Now, since $x \in \Lambda_i$ we get

$$x \in [b_{ir},e_{ir}] \qquad \text{(EQ1)}$$

So, we have $b_{ir}: \langle x,e_{ir} \rangle$ (by EQ1)
and $b_{ir}: \langle b_{ij},x \rangle$ (by Left)

Hence, $b_{ir}: \langle b_{ij},x,e_{ir} \rangle$
(by transitivity of "less than")

I.e. $x \in [b_{ij},e_{ir}]$.

$=$

We still have to show that $j \in \beta(i)$.
Assume the opposite, i.e. that
$\exists$ $k \neq i,j$ with $b_{ik} \neq b_{ij}$ such that $b_{ik} \in (b_{ij},e_{ik})$

Remembering how b_{ij} was defined, we get $x: \langle b_{ik},b_{ij} \rangle$.
But then, $x \notin [b_{ik},e_{ik}] \Rightarrow x \notin \Lambda_i$ which is a condradiction.

$\square$

So, we showed that $\Lambda_i \subseteq \bigcup_{j \in \beta(i)} [b_{ij},\text{Right}(b_{ij};E_i)]$

<u>2. $\bigcup_{j \in \beta(i)} [b_{ij},\text{Right}(b_{ij};E_i)] \subseteq \Lambda_i$</u>

Since $j \in \beta(i)$ we get (by def. of β)

$$[b_{ij},e_{im}] \subseteq [b_{im},e_{im}] \quad \forall \; m \neq i \qquad \text{(EQ3)}$$

Also, by definition of Right:

$$[b_{ij},\text{Right}(b_{ij};E_i)] \subseteq [b_{ij},e_{im}] \quad \forall \; m \qquad \text{(EQ4)}$$

by (EQ3) and (EQ4) we get

$$[b_{ij},\text{Right}(b_{ij};E_i)] \subseteq [b_{im},e_{im}] \quad \forall \; m$$

I.e. $[b_{ij}, \mathrm{Right}(b_{ij};E_i)] \subseteq \Lambda_i$
and this holds for all $j \in \beta(i)$
Hence $\bigcup_{j \in \beta(i)} [b_{ij}, \mathrm{Right}(b_{ij};E_i)] \subseteq \Lambda_i$

$\square$

(1) and (2) prove part a of the Theorem.

Part b Proof in full paper

Theorem 4.2 alone would not be so useful if the intervals were overlapping. Fortunately, the next theorem shows that this is not the case.

Theorem 4.3

(a) Let $A_{im} = [b_{im}, \mathrm{Right}(b_{im};E_i)]$
for $m=j,k \in \beta(i)$
Then either $A_{ij} = A_{ik}$ or $A_{ij} \cap A_{ik} = \emptyset$

(b) Let $\Gamma_{im} = [e_{im}, \mathrm{Right}(e_{im};B_i)]$
for $m=j,k \in \varepsilon(i)$
Then either $\Gamma_{ij} = \Gamma_{ik}$ or $\Gamma_{ij} \cap \Gamma_{ik} = \emptyset$.

For proof see Appendix A1

5. O(logn) parallel time Algorithms for Intersection and Union.

We use a total of $O(n^3)$ processors, allocated in groups of n^2 per cycle. The processors of cycle $\overline{K}_i$ are allocated as follows: A distinct group, G_{ij}, of n processors, one for each pair $\overline{K}_i$, $\overline{K}_j$ $(j \neq i)$.

1. Initialization

Initially, we let a specific processor of each group G_{ij} to compute (in parallel for each i and j) the two points b_{ij}, e_{ij} of the intersection of the two circumferences C_i, C_j. The b_{ij}, e_{ij} are computed by finding their angles (with respect to a direction $\vartheta=0^{\circ}$ independent of the discs)as seen from the center of disk $\overline{K}_i$. Let these angles be $\vartheta(b_{ij})$, $\vartheta(e_{ij})$ as in Figure 3.

This computation takes a constant number of parallel steps.

2. Computing $\beta(i)$, $\varepsilon(i)$

Initially, a distinct processor of the group G_{ij} initializes a shared variable, $check_{ij}$, to true. Then, each group G_{ij} of processors checks in parallel whether j belongs to $\beta(i)$.

That is, each processor $p_k (k=1,\dots,n)$ of the group G_{ij}, checks in parallel whether $b_{ij}:\langle e_{ik}, b_{ik}\rangle$ for $b_{ij} \neq b_{ik}$. Processors p_k record violations of $b_{ij}:\langle e_{ik}, b_{ik}\rangle$ by setting a local bit B_k to false. (If p_k records no violation, then B_k is set to true).

Finally, the logical AND of the B_k is computed in $check_{ij}$. This takes O(logn) time, in the CREW PRAM model, and O(1) time in the CRCW-PRAM. A similar procedure is followed for the case of computing $\varepsilon(i)$. Processors p_k now record violations of $e_{ij}:\langle b_{ik}, e_{ik}\rangle$ and again a logical AND has to be performed in parallel.

3. Finding Λ_i (and V_i^c)

Each group G_{ij} reads $check_{ij}$. Conditioned on $check_{ij}$=true, G_{ij} constructs the arc of C_i, with start b_{ij}, which contributes to Λ_i (with start e_{ij}, if it contributes to V_i^c) as follows:

Each processor p_k in G_{ij} computes the angles

$$\alpha_{ik} = \vartheta(b_{ik}) - \vartheta(b_{ij})$$

$$\gamma_{ik} = \vartheta(e_{ik}) - \vartheta(b_{ij})$$

Then the group G_{ij} computes the $x_{ij} = \min\{\gamma_{ik},\ k \neq i\}$ and $y_{ij} = \min\{\alpha_{ik},\ k \neq i\}$. This takes O(logn) time in the CREW PRAM model. The contribution to Λ_i is $[b_{ij}, x_{ij}]$ and to V_i^c it is $[e_{ij}, y_{ij}]$. (In the CRCW PRAM model, step 3 would take O(loglogn) time).

Note

The algorithm just presented constructs the contour surrounding the intersection I (if $I \neq \emptyset$) and the union Σ of the n circular disks. In fact, each x_{ij} is a b_{kh} of another C_k and one can "walk" on the boundary of I (or Σ) in this way. The areas of I and Σ are oriented sums of the areas defined by the arcs in $\bigcup_i \Lambda_i$

and $\bigcup_i V_i^c$ and some set of fixed coordinate axes and thus can be computed in O(logn) time in the CREW PRAM. The data structure constructed by the algorithm implies an O(1) time for answering queries of the form "given the point (x,y) does it belong to the boundary of I or the boundary of Σ".

Note also that the algorithm just presented is of O(loglogn) time in the CRCW-PRAM model, since logical AND takes O(1) time there and "max" takes O(loglogn) time.

6. An argument for optimality of the CREW-PRAM algorithms.

By using our algorithm we can determine the area of the union of n circular disks in O(logn) time. Given a sequence of n numbers in the field of reals or in any subfield of relatively long integers, call them $x_1,\ldots,x_n$, we construct the disks of radii $\sqrt{x_1},\ldots,\sqrt{x_n}$ and center coordinates $(3k+1 \max\sqrt{x_i},0)$ (which, obviously, are non-intersecting). The area of their union is then $\pi\cdot \mathrm{sum}(x_1,\ldots,x_n)$. Any algorithm to compute the area of the union in o(logn) time would also compute $\mathrm{sum}(x_1,\ldots,x_n)$ in o(logn) time. But this contradicts to a result of I. Parberry (see [Pa, 1986]).

7. The number of arcs in the boundary of the intersection.

Theorem 7.1 The boundary of the intersection I of n circular disks in the plane consists of at most 2n-2 arcs, for any $n \geq 2$.

Proof

We need three helpful lemmas:

Lemma 7.1 Let arcs x, y_1 and y_2 be in the boundary B(I) of the intersection I of $n \geq 2$ circles. Let x be between y_1 and y_2 and let it intersect with y_1 at point A and with y_2 at point B. Let y_1 and y_2 belong to the circumference C(Y) of disk Y and x to the circumference C(X) of disk X.

Then, $C(X)$ has only one arc (i.e. x) in $B(I)$.

For proof of Lemma 7.1 see Appendix A2

Lemma 7.2 Let arcs x_1, x_2 (of $C(X)$ of disk X) and arcs y_1, y_2 (of the circumference $C(Y)$ of disk Y) belong to the boundary $B(I)$ of the intersection I. Then, if we start from x_1 and order arcs on $B(I)$ by the order we meet them when proceeding on $B(I)$ counterclockwise, either both y_1, y_2 will appear before x_2, or both will appear after x_2.

For proof of Lemma 7.2 see Appendix A3

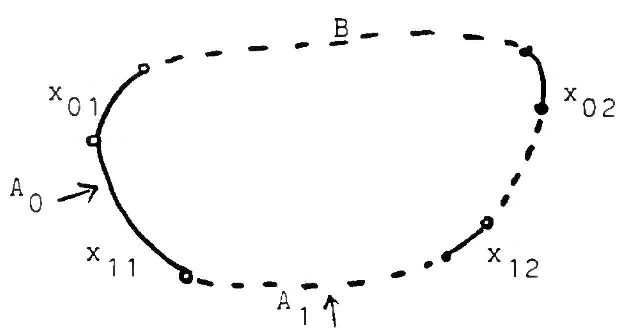
Lemma 7.3 For any $n \geq 2$, there exists a disk (out of the n disks participating in $B(I)$) which has only one arc on $B(I)$, provided that I does not collapse to a point.

Proof

Assume, for sake of contradiction, that all circles contribute at least two arcs each in $B(I)$. Consider two arcs, x_{01},x_{02} of such a circle, X_0, and let us look at the one of the two connected pieces of $B(I)$ which joins x_{01} with x_{02} (call it A_0).

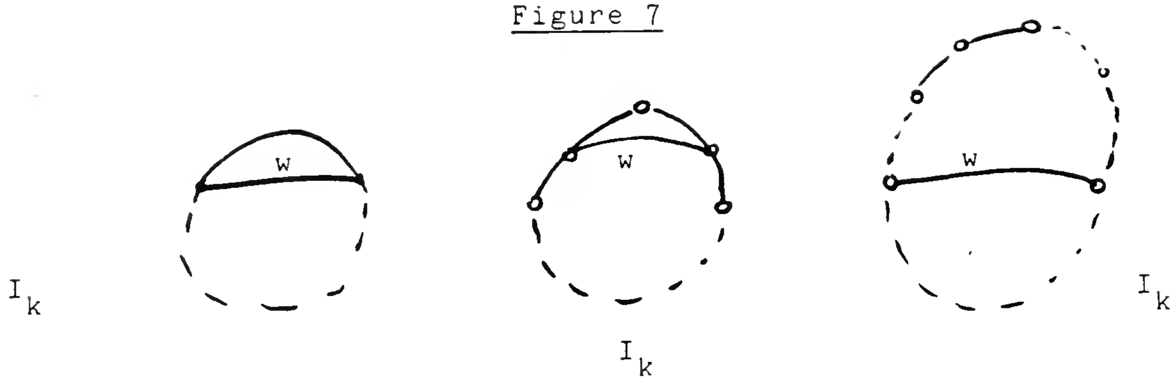
Let X_1 be the circle whose arc, x_{11}, connects to x_{01} of X_0 and belongs on A_0. Then (by Lemma 2) any other arc x_{12} of X_1 must lie on A_0 too. Let x_{12} be the arc of X_1 on A_0 such that no other arc of X_1 is between x_{11} and x_{12} on A_0. Let A_1 be the piece of A_0 between x_{11} and x_{12}.

Figure 6



I_{k+1}. We distinguish 3 situations:

Figure 7



Case I | Case II | Case III

Case I The number of vertices of $B(I_k)$ left out by w is zero. In this case, w increases the number of arcs by 2, since it replaces a single arc of a single circle by 3 arcs.

Case II The number of vertices of $B(I_k)$ left out by w is 1. In that case, w increases the number of arcs by 1.

Case III The number of vertices of $B(I_k)$, left out by w, is >1. In that case, w does not increase the number of arcs in $B(I_{k+1})$.

So, the biggest increase is that of case I. Hence, the number of arcs, β_{k+1}, in $B(I_{k+1})$ becomes (in any case)

$$\beta_{k+1} \leq \beta_k + 2 \leq 2k-2+2$$
$$= 2(k+1)-2.$$

So, the theorem holds for $n=k+1$, hence for all $n>2$.

$\square$

Figure 1

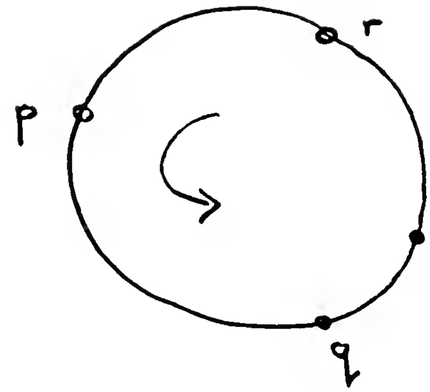


Figure 2

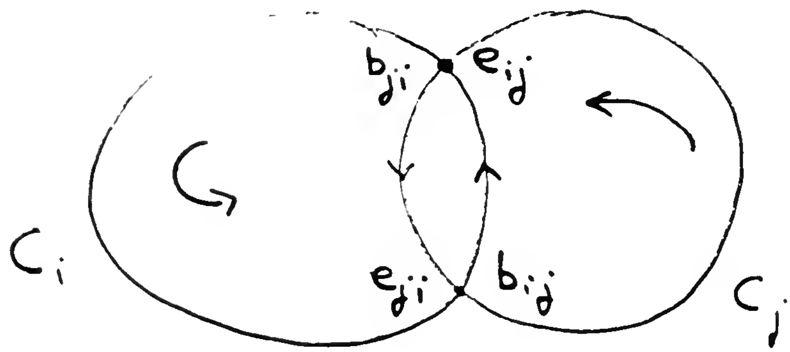
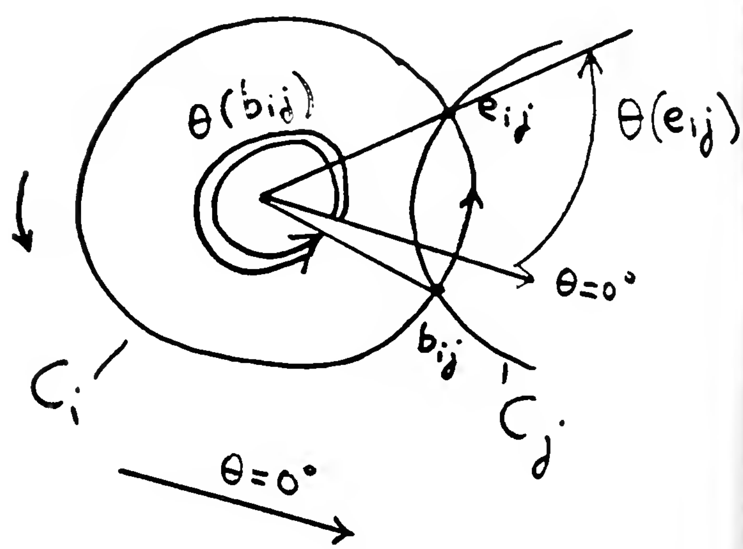


Figure 3



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APPENDIX A1

Part a

Let e_{ir}=Right $(b_{ij};E_i)$ and
e_{is}=Right $(b_{ik};E_i)$

Assume (for contradiction) that $A_{ij} \cap A_{ik} \neq \emptyset$ and $A_{ik} \neq A_{ij}$. This means that

$$[b_{ij}, e_{ir}] \cap [b_{ik}, e_{is}] \neq \emptyset$$

By the definition of e_{ir} we must have

$$b_{ij} : \langle e_{ir}, e_{is} \rangle \qquad \text{(EQ5)}$$

Also, similarly, $b_{ik} : \langle e_{is}, e_{ir} \rangle$ (EQ6)

Since $j \in \beta(i)$ we get $b_{ij} : \langle e_{im}, b_{im} \rangle \ \forall m \neq i,j$ (EQ7)

Also, since $k \in \beta(i)$ we get $b_{ik} : \langle e_{im}, b_{im} \rangle \ \forall m \neq i,k$ (EQ8)

From (EQ5), (EQ6) we get

$$b_{ij} : \langle e_{ir}, b_{ik}, e_{is} \rangle \qquad \text{(EQ9)}$$

From (EQ7) we have $b_{ij} : \langle e_{ik}, b_{ik} \rangle$ (EQ10)

So, there are two possibilities for the position of e_{ik}.

Either $\quad b_{ij} : \langle e_{ik}, e_{ir}, b_{ik} \rangle$

or $\quad b_{ij} : \langle e_{ir}, e_{ik}, b_{ik} \rangle$

The first possibility can not happen because of the definition of e_{ir}.

We conclude that $b_{ij} : \langle e_{ir}, e_{ik}, b_{ik} \rangle$ (EQ11)

From (EQ8) we get $b_{ik} : \langle e_{ij}, b_{ij} \rangle$ By similar reasoning we conclude that

$$b_{ik} : \langle e_{is}, e_{ij}, b_{ij} \rangle \qquad \text{(EQ12)}$$

But then, clearly, $[b_{ij}, e_{ir}]$ and $[b_{ik}, e_{is}]$cannot intersect which is a contradiction

$\square$

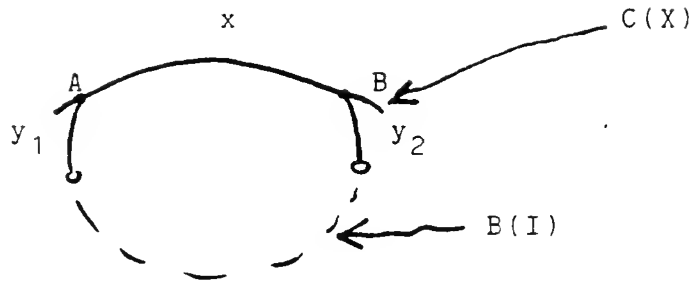
The proof of Part 2 is very similar

QED

APPENDIX A2

Proof of Lemma 7.1

Figure 4



Set I is a convex set. Hence, the rest of the boundary of I (except x) is totally within circle X

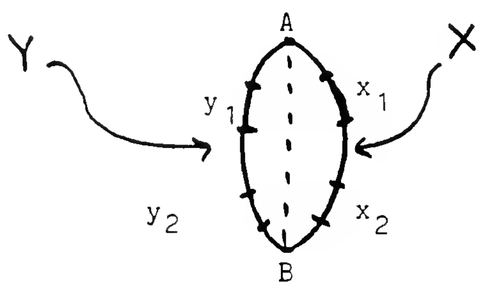
□

APPENDIX A3

Proof of Lemma 7.2

Consider the intersection I(X,Y) of disks X,Y.

Figure 5



Let A,B be the points of intersection of C(X), C(Y). The line AB totally separates all the points of the arc $\widehat{AB}$ of C(Y) from all the points of the arc $\widehat{AB}$ of C(X) (except, of course for A,B). If the arcs x_1,x_2,y_1,y_2 were "intermixed" in B(I) then no straight line would separate both x_1,x_2 from both y_1,y_2.

□

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